

A SPECTRAL THEORY FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

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ABSTRACT. The main objective of this paper is to develop a notion of joint spectrum for complex solvable Lie algebras of operators acting on a Banach space, which generalizes the Taylor joint spectrum (T.J.S.) for several commuting operators.

1. INTRODUCTION

We briefly recall the definition of the Taylor joint spectrum. Let $\wedge(\mathbb{C}^n)$ be the complex exterior algebra on n generators e_1, \dots, e_n with multiplication denoted by \wedge . Let E be a Banach space and $a = (a_1, \dots, a_n)$ a mutually commuting n -tuple of bounded linear operators on E (m.c.o.). Define $\wedge_k^n(E) = \wedge_k(\mathbb{C}^n) \otimes_{\mathbb{C}} E$ and D_{k-1} by:

$$D_{k-1}: \wedge_k^n(E) \rightarrow \wedge_{k-1}^n(E)$$

$$D_{k-1}(x \otimes e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{j=1}^k (-1)^{j+1} x \cdot a_{i_j} \otimes e_{i_1} \wedge \dots \wedge \tilde{e}_{i_j} \wedge \dots \wedge e_{i_k},$$

where $\tilde{}$ means deletion and $k \geq 1$. Also define $D_k = 0$ for $k \leq 0$.

It is easy to prove that $D_k D_{k+1} = 0$ for all k , that is, $(\wedge_k^n(E), D_k)_{k \in \mathbb{Z}}$ is a chain complex, which is called the Koszul complex associated to a and E and it is denoted by $R(E, a)$. The n -tuple a is said to be invertible or nonsingular on E , if $R(E, a)$ is exact, i.e., $\text{Ker}(D_k) = \text{Ran}(D_{k+1})$ for all k . The Taylor spectrum of a on E is $Sp(a, E) = \{\lambda \in \mathbb{C}^n : a - \lambda \text{ is not invertible}\}$.

Unfortunately, this definition depends very strongly on a_1, \dots, a_n and not on the vector subspace of $L(E)$ generated by them, i.e., $\langle a \rangle$.

As we consider Lie algebras, which involve naturally geometry, we are interested in a geometrical approach to the spectrum which depends on L rather than on a particular set of operators.

This is done in section 2. Given a solvable Lie subalgebra of $L(E)$, L , we associate to it a set in L^* , $Sp(L, E)$.

This object has the classical properties, i.e., $Sp(L, E)$ is a compact set, if L' is an ideal of L , then $Sp(L', E)$ is the projection of $Sp(L, E)$ on L'^* , and finally, $Sp(L, E)$ is a non empty set.

Besides, it satisfy other interesting properties.

If $x \in L^2$, then $Sp(x) = 0$. Furthermore, if L is nilpotent, one has the inclusion

$$Sp(L, E) \subseteq \{f \in [L, L]^\perp : \forall x \in L, \|f(x)\| \leq \|x\|\}.$$

However, the spectral mapping property is ill behaved.

2. THE JOINT SPECTRUM FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

First of all we establish a proposition which will be used in the definition of $Sp(L, E)$.

From now on L denotes a complex finite dimensional solvable Lie algebra and $U(L)$ its enveloping algebra.

Let f belong to L^* such that $f([L, L]) = 0$, i.e., f is a character of L . Then f defines a one dimensional representation of L denoted by $\mathbb{C}(f)$. Let $\epsilon(f)$ be the augmentation of $U(L)$ defined by f :

$$\begin{aligned}\epsilon(f): U(L) &\rightarrow \mathbb{C}(f), \\ \epsilon(f)(x) &= f(x) \quad (x \in L).\end{aligned}$$

Let us consider the pair of spaces and maps $V(L) = (U(L) \otimes \wedge L, \bar{d}_{p-1})$, where \bar{d}_{p-1} is the map defined by:

$$\bar{d}_{p-1}: U(L) \otimes \wedge^p L \rightarrow U(L) \otimes \wedge^{p-1} L.$$

If $p \geq 1$

$$\begin{aligned}\bar{d}_{p-1}\langle x_{i_1} \cdots x_{i_p} \rangle &= \sum_{k=1}^p (-1)^{k+1} (x_{i_k} - f(x_{i_k})) \langle x_{i_1} \wedge \cdots \wedge \hat{x}_{i_k} \wedge \cdots \wedge x_{i_p} \rangle \\ &+ \sum_{1 \leq k < l \leq p} (-1)^{k+l} \langle [x_{i_k}, x_{i_l}] \wedge x_{i_1} \wedge \cdots \wedge \hat{x}_{i_k} \wedge \cdots \wedge \hat{x}_{i_l} \wedge \cdots \wedge x_{i_p} \rangle\end{aligned}$$

where $\hat{}$ means deletion. If $p \leq 0$, we also define $\bar{d}_p = 0$.

Proposition 2.1. *The pair of spaces and maps $V(L)$ is a chain complex. Furthermore, with the augmentation $\epsilon(f)$, the complex $V(L)$ is a $U(L)$ -free resolution of $\mathbb{C}(f)$ as a left $U(L)$ -module.*

We omit the proof of Proposition 2.1 because it is a straightforward generalization of [3, Chapter XIII, Theorem 7.1].

Let L be as usual. From now on E denotes a Banach space on which L acts as right continuous operators, i.e., L is a Lie subalgebra of $L(E)$ with the opposite product. Then, by [3, Chapter XIII, Section 1], E is a right $U(L)$ module.

If f is a character of L , by Proposition 2.1 and elementary homological algebra, the q -homology space of the complex $(E \otimes \wedge L, d(f))$ is $Tor_q^{U(L)}(E, \mathbb{C}(f)) = H_q(L, E \otimes \mathbb{C}(f))$.

We now state our definition.

Definition 2.2. *Let L and E be as above. The set $\{f \in L^* : f(L^2) = 0, H_*(L, E \otimes \mathbb{C}(f)) \text{ is non-zero}\}$, is the spectrum of L acting on E , and it is denoted by $Sp(L, E)$.*

By Proposition 2.1 and Definition 2.2 it is clear that if L is a commutative algebra, then $Sp(L, E)$ reduces to the Taylor joint spectrum.

Let's see an example. Let $(E, \|\cdot\|)$ be $(\mathbb{C}^2, \|\cdot\|_2)$ and a, b the operators

$$a = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

It is easy to prove that $[b, a] = b$, and then the vector space $\mathbb{C}(b) \oplus \mathbb{C}(a) = L$ is a solvable Lie subalgebra of $L(\mathbb{C}^2)$.

Using Definition 2.2, a standard calculation shows that $Sp(L, E) = \{f \in \mathbb{C}^{2*} : f(b) = 0, f(a) = \frac{1}{2}, f(a) = \frac{-3}{2}\}$.

Observe that, $\|a\| = \frac{1}{2}$; however, $Sp(L, E)$ is not contained in $\{f \in \mathbb{C}^{2*} : \forall x \in \mathbb{C}^2 \mid f(x) \leq \|x\|\}$.

3. FUNDAMENTAL PROPERTIES OF THE SEPCTRUM

In this section we shall prove that the most important properties of spectral theory are satisfied by our spectrum.

Theorem 3.1. *Let L and E be as usual. Then $Sp(L, E)$ is a compact set of L^* .*

Proof. Let us consider the family of spaces and maps $(E \otimes \wedge^i L, d_{i-1}(f))$, where $f \in L^{2\perp}$ and $L^{2\perp} = \{f \in L^* : f(L^2) = 0\}$. This family is a parameterized chain complex on $L^{2\perp}$. By Taylor [6, Theorem 2.1] the set $\{f \in L^{2\perp} : (E \otimes \wedge^i L, d_{i-1}(f)) \text{ is exact}\} = Sp(L, E)^c$ is an open set in $L^{2\perp}$. Then, $Sp(L, E)$ is closed in $L^{2\perp}$ and hence in L^* .

To verify that $Sp(L, E)$ is a compact set, we consider a basis of L^2 and we extend it to a basis of L , $\{X_i\}_{1 \leq i \leq n}$. If $K = \dim L^2$, $n = \dim L$ and $i \geq K + 1$, then let L_i be the ideal generated by $\{X_j\}_{1 \leq j \leq n, j \neq i}$.

Ler f be a character of L and represent it in the dual basis of $\{X_i\}_{1 \leq i \leq n}$, $\{f_i\}_{1 \leq i \leq n}$, $f = \sum_{i=K+1}^n \xi_i f_i$. For each i , there is a positive number r_i such that if $\xi_i \geq r_i$,

$$Tor_p^{U(L)}(E, \mathbb{C}(f)) = H_p(E \otimes \wedge^i, d_{i-1}(f)) = 0 \quad \forall p.$$

To prove our last statement, we shall construct an homotopy operator for the chain complex $(E \otimes \wedge^p L, d_{p-1}(f))$, $(f(L^2) = 0)$.

First of all we observe that

$$E \otimes \wedge^p L = (E \otimes \wedge^p L_i) \oplus (E \otimes \wedge^{p-1} L_i) \wedge \langle X_i \rangle.$$

As L_i is an ideal of L , $d_{p-1}(E \otimes \wedge^p L_i) \subseteq E \otimes \wedge^{p-1} L_i$. On the other hand, there is a bounded operator L_{p-1} such that

$$d_{p-1}(f)(a \wedge \langle X_i \rangle) = (d_{p-1}(f)(a)) \wedge \langle X_i \rangle + (-1)^p L_{p-1}(a), a \in E \otimes \wedge^{p-1} L_i.$$

It is easy to prove that, for each p , there is a basis of $\wedge^p L_i$, $\{V_j^p\}$ $1 \leq j \leq \dim \wedge^p L_i$, such that if we decompose

$$E \otimes \wedge^p L_i = \bigoplus_{1 \leq j \leq \dim \wedge^p L_i} E \langle V_j \rangle,$$

then L_p has the following form:

$$\begin{aligned}
L_{p_{ij}} &= \alpha_{ij}^p && \text{for } i < j, \\
L_{p_{ij}} &= X_i - \xi_i + \alpha_{jj}^p, && \text{for } i = j \\
L_{p_{ij}} &= 0, && \text{for } i > j,
\end{aligned}$$

where $\alpha_{ij}^p \in \mathbb{C}$.

Besides, let K_p be a positive real number such that

$$\bigcup_{1 \leq j \leq \dim \wedge^p L_i} Sp(X_i + \alpha_{jj}^p) \subseteq B[0, K_p]$$

and $N_i = \max_{0 \leq p \leq n-1} \{K_p\}$. Then, as L_p has a triangular form, a standard calculation shows that L_p is a topological isomorphism of Banach spaces if $\xi_i \geq N_i$.

Outside $B[0, N_i]$ we construct our homotopy operator

$$\begin{aligned}
Sp: E \otimes \wedge^p L &\rightarrow E \otimes \wedge^{p+1} L, \\
Sp|_{E \otimes \wedge^{p-1} L_i \wedge \langle X_i \rangle} &\equiv 0, \\
Sp: E \otimes \wedge^p L_i &\rightarrow E \otimes \wedge^p L_i \wedge \langle X_i \rangle, \\
Sp &= (-1)^{p+1} L_p^{-1} \wedge \langle X_i \rangle.
\end{aligned}$$

From the definition of L_p we have the following identity:

$$(-1)^{p+2} S_{p-1} d_{p-1}(f) L_p = d_{p-1}(f) \wedge \langle X_i \rangle.$$

The above identity and a standard calculation prove that Sp is a homotopy operator, i.e., $d_p S_p + S_{p-1} d_{p-1} = I$. Therefore, $Sp(L, E)$ is a compact set. \square

Theorem 3.2. (*Projection Property*) *Let L and E be as usual, and let I be an ideal of L . Let π be the projection map from L^* onto I^* , then*

$$Sp(I, E) = \pi(Sp(L, E)).$$

Proof. By [2, Section 5.3, Corollaire 3], there is a Jordan Hölder sequence of L such that I is one of its terms. Then, by means of an induction argument, we can assume $\dim(L/I) = 1$.

Let's consider the connected simply connected complex Lie group $G(L)$ such that its algebra is L , [5, Part II-Lie Groups, Chapter 5].

Let Ad^* be the coadjoint representation of $G(L)$ in L^* : $Ad^*(g)(f) = f Ad(g^{-1})$, where $g \in G(L)$, $f \in L^*$ and Ad is the adjoint representation of $G(L)$ in L .

Let f belong to $Sp(I, E)$. Then, as I is an ideal of L , by [7, Theorem 2.13.4], $Ad^*(g)(f)$ belongs to I^* . Besides, it is a character of I . Then, one can restrict the coadjoint action of $G(L)$ to I^* . Moreover, $Sp(I, E)$ is invariant under the coadjoint action of $G(L)$ in I^* , i.e., if $f \in Sp(I, E)$, then $Ad^*(g)(f) \in Sp(I, E)$, $\forall g \in G(L)$.

To this end, it is enough to prove that

$$(I) \quad Tor_*^{U(I)}(E, \mathbb{C}(f)) \cong Tor_*^{U(I)}(E, \mathbb{C}(h))$$

where $h = Ad^*(g)(f)$, $g \in G(L)$.

Let Γ be the ring $U(I)$ and φ the ring morphism

$$\varphi = U(\text{Ad}g): U(I) \rightarrow U(I).$$

Let's consider the augmentation modules $(\mathbb{C}(f), E(f))$ and $(\mathbb{C}(h), E(h))$.

Then, a standard calculation shows that the hypothesis of [3, Chapter VIII, Theorem 3.1] are satisfied, which implies (I).

Thus, if $f \in Sp(I, E)$, the orbit $G(L) \cdot f \subseteq Sp(I, E)$. However, $Sp(I, E)$ is a compact set of I^* .

As the only bounded orbits for an action of a complex connected Lie group on a vector space are points, then $G(L) \cdot f = f$.

Let \bar{f} be an extension of f to L^* , and consider $\alpha = G(L) \cdot \bar{f}$, the orbit of \bar{f} under the coadjoint action of $G(L)$ in L^* .

As $G(L) \cdot f = f$, then as an analytic manifold

$$(II) \quad \dim \alpha \leq 1.$$

Now suppose \bar{f} is not a character of L , i.e., $\bar{f}(L^2) \neq 0$.

Let L^\perp be the following set: $L^\perp = \{x \in L : \bar{f}([X, L]) = 0\}$, and let n be the dimension of L .

As I is an ideal of dimension $n-1$, $f(I^2) = 0$ and $f(L^2) \neq 0$, by [2, Section 5.3, Corollaire 3], [1, Chapitre IV, Section 4.1, Proposition 4.1.1] and by [4, Chapter 1, 1.2.8], we have $L^\perp \subset I$ and $\dim L^\perp = n-2$.

Let's consider the analytic subgroup of $G(L)$ such that its Lie algebra is L^\perp .

As the Lie algebra of the subgroup $G(L)_{\bar{f}} = \{g \in G(L) : \text{Ad}^*(g)(\bar{f}) = \bar{f}\}$ is L^\perp , the connected component of the identity of $G(L)_{\bar{f}}$ is $G(L^\perp)$.

Howewer, by [7, Lemma 2.9.2, Theorem 2.9.7] $\alpha = G(L) \cdot \bar{f}$ satisfies the following properties: $\alpha \cong G(L)/G(L)_{\bar{f}}$, and $\dim \alpha = \dim G(L) - \dim G(L)_{\bar{f}} = \dim G(L) - \dim G(L^\perp) = \dim L - \dim L^\perp = 2$, which contradicts (II).

Then, \bar{f} is a character of L .

Thus, any extension \bar{f} of f in $Sp(L, E)$ is a character of L .

However, as in [6], there is a short exact sequence of complexes

$$0 \rightarrow (\wedge^* I \otimes E, d(f)) \rightarrow (\wedge^* L \otimes E, d(\bar{f})) \rightarrow (\wedge^* I \otimes E, d(f)) \rightarrow 0.$$

As $U(I)$ is a subring with unit of $U(L)$ and the complex involved in Definition 2.2 differs from the one of [6] by a constant term, Taylor's joint argument of [6, Lemma 3.1] still applies and then $Sp(I, E) = \pi(Sp(L, E))$. \square

As a consequence of Theorem 3.2 we have the following result.

Theorem 3.3. *Let L and E be as usual. Then $Sp(L, E)$ is not void.*

4. SOME CONSEQUENCES

In this section we shall see some consequences of the main theorems.

Let E be a Banach space and L a complex finite dimensional solvable Lie algebra acting on E as bounded operators.

One of the well known properties of Taylor joint spectrum for an n -tuple of m.c.o. acting on E is $Sp(a, E) \subseteq \Pi B[0, \|a_i\|]$. In the noncommutative case, as we have seen in section 2, this property fails.

However, if the Lie algebra is nilpotent, it is still true.

Proposition 4.1. *Let L be a nilpotent Lie algebra which acts as bounded operators on a Banach space E .*

Then, $Sp(L, E) \subseteq \{f \in L^ : |f(x)| \leq \|x\|, x \in L\}$.*

Proof. We proceed by induction on $\dim L$. If $\dim L = 1$, we have nothing to verify.

We suppose that the proposition holds for every nilpotent Lie algebra L' such that $\dim L' < n$.

If $\dim L = n$, by [2, Section 4.1, Proposition 1], there is a Jordan Hölder sequence $S = (L_i)_{0 \leq i \leq n}$, such that $[L, L_i] \subseteq L_{i-1}$.

Let $\{X_i\}_{1 \leq i \leq n}$ be a basis of L such that $\{X_j\}_{1 \leq j \leq i}$ generates L_i .

Let L'_{n-1} be the vector subspace generated by $\{X_i\}_{1 \leq i \leq n, i \neq n-1}$. As $[L, L'_{n-1}] \subseteq L_{n-2} \subseteq L'_{n-1}$, L'_{n-1} is an ideal. Besides, $L_{n-1} + L'_{n-1} = L$.

Then, by means of Theorem 3.2 and the inductive hypothesis, we complete the inductive argument and the proposition. \square

Now, we deal with some consequences of the projection property.

Proposition 4.2. *Let L and E be as usual. If I is an ideal contained in L^2 , then $Sp(I, E) = 0$. In particular, $Sp(L^2, E) = 0$.*

Proof. By the projection property, $Sp(I, E) = \pi(Sp(L, E))$, where π is the projection from L^* onto I^* . However, as $Sp(L, E)$ is a subset of characters of L , $f|_I = 0$, if $I \subseteq L^2$. \square

Proposition 4.3. *Let L and E be as in Proposition 4.1. If $Sp(L, E) = 0$, then $Sp(x) = 0, \forall x \in L$.*

Proof. By means of and inductive argument, the ideals L_{n-1}, L'_{n-1} of Proposition 4.1 and Theorem 3.2, we conclude the proof. \square

Proposition 4.4. *Let L and E be as usual. Then, if $x \in L^2$, $Sp(x) = 0$.*

Proof. First of all, recall that if L is a solvable Lie algebra, then L^2 is a nilpotent Lie algebra. Therefore, by Proposition 4.2, $Sp(L^2, E) = 0$, and by Proposition 4.3, $Sp(x) = 0 \forall x \in L^2$. \square

Remark 4.5. Note that the example of section 2 shows that the projection property fails for subspaces which are not ideals (take $I = \langle x \rangle$). Clearly this implies that the spectral mapping theorem also fails in the noncommutative case.

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